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NON-ABELIAN PLANE-WAVES IN THE QUARK-GLUON PLASMA

Jean-Paul BLAIZOT* and Edmond IANCU
Service de Physique Théorique†*, CE-Saclay*
91191 Gif-sur-Yvette, France

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Abstract

We present new, non-abelian, solutions to the equations of motion which describe the collective excitations of a quark-gluon plasma at high temperature. These solutions correspond to longitudinal and transverse plane-waves propagating through the plasma.

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*CNRS

†Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique

1 Introduction. General equations

The long wavelength excitations of a quark-gluon plasma are collective excitations which are described by nonlinear equations generalizing the classical Yang-Mills equations in the vacuum. Most studies have been limited so far to the weak field limit, where the equations become linear and the excitations reduce to abelian-like plasma waves[1], very much similar to the electromagnetic waves in ordinary plasmas[2]. The purpose of this paper is to present new, truly non-abelian, plane-wave solutions that we have obtained recently. At leading order in the gauge coupling g, (we assume $g \ll 1$ in the high temperature, deconfined, QCD plasma), the collective dynamics is entirely described by a set of equations for the gauge mean fields $A_a^\mu(x)$ which describe the long wavelength ($\lambda \sim 1/gT$) and low frequency ($\omega \sim gT$) excitations (T denotes the temperature)[3, 4]. (Throughout this work, the greek indices refer to Minkovski space, while the latin ones are color indices for the adjoint representation of the gauge group $SU(N)$.) The equations satisfied by $A_a^\mu(x)$ are

$$[\, D^\nu,\, F_{\nu\mu}(x)\,]_a \;=\; j_a^\mu(x), \tag{1.1}$$

where $D^\mu = \partial^\mu + igA^\mu(x)$, $A^\mu = A_a^\mu T^a$, and $F^{\mu\nu} = [D^\mu, D^\nu]/(ig) = F_a^{\mu\nu} T^a$. In the right hand side, j_a^μ is the *induced current* which describes the response of the plasma to the color gauge fields A_a^μ. (We do not consider here any external color source.) Its expression is[3, 5]

$$j_a^\mu(x) \;=\; 3\,\omega_p^2 \int \frac{d\Omega}{4\pi}\, v^\mu\, W_a^0(x; v), \tag{1.2}$$

where $\omega_p^2 \equiv (2N + N_{\rm f})g^2T^2/18$ is the *plasma frequency*, $v^\mu \equiv (1, \mathbf{v})$, $\mathbf{v} \equiv \mathbf{k}/k$, $k \equiv |\mathbf{k}|$, and the integral $\int d\Omega$ runs over all the directions of the unit vector $\mathbf{v}$. The functions $W^\mu \equiv W_a^\mu T^a$ are defined as the solutions to

$$[v \cdot D_x,\, W^\mu(x; v)] \;=\; F^{\mu\rho}(x)\, v_\rho. \tag{1.3}$$

For retarded conditions (i.e., $A_a^\mu(x) \to 0$ as $x_0 \to -\infty$)

$$W_a^\mu(x; v) \;=\; \int_0^\infty du\, U_{ab}(x, x - vu)\, F_b^{\mu\rho}(x - vu)\, v_\rho, \tag{1.4}$$

where $U(x,y)$ is the parallel transporter along the straight line γ joining x and y:

$$U(x,y) = P\exp\{-ig\int_\gamma dz^\mu A_\mu(z)\}. \tag{1.5}$$

The energy-momentum tensor of an arbitrary gauge field configuration in the plasma has been recently computed[5], with the following results for the energy density

$$T^{00}(x) \;=\; \frac{1}{2}\Big(\mathbf{E}^a(x)\cdot\mathbf{E}^a(x) \,+\, \mathbf{B}^a(x)\cdot\mathbf{B}^a(x)\Big) \,+\, \frac{3}{2}\,\omega_p^2\int\frac{d\Omega}{4\pi}\,W_a^0(x;v)\,W_a^0(x;v), \tag{1.6}$$

and for the energy flux density, or Poynting vector, $S^i \equiv T^{i0}$,

$$\mathbf{S}(x) \;=\; \mathbf{E}^a(x)\times\mathbf{B}^a(x) \,+\, \frac{3}{2}\,\omega_p^2\int\frac{d\Omega}{4\pi}\;\mathbf{v}\;W_a^0(x;v)\,W_a^0(x;v). \tag{1.7}$$

The non-abelian field strengths are, as usually, $E^i \equiv F^{i0}$ and $B^i_a \equiv -(1/2)\epsilon^{ijk}\,F^{jk}_a$. The first terms in the r.h.s. of eqs. (1.6)–(1.7) represent the standard Yang-Mills contributions; the terms depending on $W^0_a(x;v)$ are related to the color polarizability of the plasma. In the absence of external sources, the energy conservation requires that

$$\partial_0\,T^{00}(x) \,+\, \partial_i\,S^i(x) \;=\; 0. \tag{1.8}$$

It can be easily verified by using the equations of motion (1.1)–(1.3) that this equation is indeed satisfied.

2 Plane-wave solutions

In this letter, we study particular plane-wave solutions which depend on x^μ only through the variable $z \equiv p^\mu x_\mu$, where $p^\mu \equiv (\omega,\mathbf{p})$ is a fixed, time-like, four-vector ($p^\mu p_\mu = \omega^2 - p^2 > 0$, $p \equiv |\mathbf{p}|$). Thus, we search for solutions of the form

$$A^\mu_a(x) \;=\; \mathcal{A}^\mu_a(p\cdot x), \tag{2.1}$$

in a gauge to be specified later. For the classical Yang-Mills equations in the vacuum, plane-wave solutions of this form have been investigated in Refs. [7, 8]. Another class of non-abelian plane-waves in the vacuum was considered by Coleman[9]. We note that, in

contrast to the vacuum case, the solutions of eqs. (1.1) for the high temperature plasma have direct physical relevance: they correspond to the collective color excitations of the QCD plasma.

In deriving eqs. (1.1)–(1.3), we have assumed[3] the gauge fields to be weak ($A \lesssim T$) and slowly varying ($\partial A \sim gTA$). For consistency, we therefore require that $\omega \sim p \sim gT$ and $|\mathcal{A}_a^\mu| \lesssim T$ for the plane waves (2.1). An important consequence of the Ansatz (2.1) is that the corresponding functions $W_a^\mu(x;v)$ are local and linear in the gauge fields:

$$W_a^\mu(x;v) \,=\, -\,\mathcal{A}_a^\mu(z) \,+\, \frac{p^\mu}{v\cdot p}\, v\cdot \mathcal{A}_a(z), \tag{2.2}$$

as can be verified on eq. (1.3). Then, the induced current (1.2) may be easily computed, with the following results

$$\rho_a(x) \,=\, -\,3\omega_p^2\, \alpha(\omega/p)\Big(\mathcal{A}_a^L \,-\, \frac{p}{\omega}\,\mathcal{A}_a^0\Big), \tag{2.3}$$

and

$$\mathbf{j}_a(x) \,=\, -\,3\omega_p^2\, \frac{\omega}{p}\Big\{\alpha(\omega/p)\Big(\mathcal{A}_a^L \,-\, \frac{p}{\omega}\,\mathcal{A}_a^0\Big)\,\hat{\mathbf{p}} \,-\, \beta(\omega/p)\,\boldsymbol{\mathcal{A}}_a^T\Big\}, \tag{2.4}$$

where $\mathcal{A}_a^L \equiv \hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{A}}_a$ ($\hat{\mathbf{p}} \equiv \mathbf{p}/p$), and $\boldsymbol{\mathcal{A}}_a^T \equiv \boldsymbol{\mathcal{A}}_a - \hat{\mathbf{p}}\mathcal{A}_a^L$. The functions $\alpha(u)$ and $\beta(u)$ ($u \equiv \omega/p$) appear when evaluating the angular integral in (1.2). We have ($\omega > p$)

$$\int \frac{d\Omega}{4\pi}\, \frac{v^i\, v^j}{\omega - \mathbf{v}\cdot\mathbf{p}} \,=\, \frac{\alpha(\omega/p)}{p^2}\,\hat{p}^i\,\hat{p}^j \,+\, \frac{\beta(\omega/p)}{p^2}\,(\delta^{ij} - \hat{p}^i\,\hat{p}^j), \tag{2.5}$$

with

$$\alpha(u) \equiv u\,\Big(Q(u) \,-\, 1\Big), \qquad \beta(u) \equiv \frac{u}{2}\,\Big(1 \,-\, \frac{u^2-1}{u^2}\,Q(u)\Big), \tag{2.6}$$

and $Q(u)$ is defined by

$$Q(u) \equiv \frac{u}{2}\,\ln\frac{u+1}{u-1}. \tag{2.7}$$

For $u > 1$, the functions $\alpha(u)$ and $\beta(u)$ are positive. Remark that the current (2.3)–(2.4) is not only covariantly conserved, $[D_\mu, j^\mu] = 0$, as it should for the consistency of eq. (1.1), but it also verifies $\partial_\mu j_a^\mu = p_\mu j_a^\mu = 0$.

The expressions (2.3)–(2.4) for the induced current are formally identical to those corresponding to an abelian plasma, that is, they involve only the gauge field polarization tensor, and no higher order irreducible amplitudes. This simplification represents an essential feature of the plane-wave fields that we are considering here.

For later use, we evaluate here the polarization pieces of the energy density (1.6) and of the Poynting vector (1.7) for the plane-waves (2.1). We need the integrals ($\omega > p$):

$$\int \frac{d\Omega}{4\pi}\,\frac{v^i\,v^j}{(\omega - \mathbf{v}\cdot\mathbf{p})^2} \;=\; \frac{a(\omega/p)}{p^2}\,\hat{p}^i\,\hat{p}^j \;+\; \frac{b(\omega/p)}{p^2}\,(\delta^{ij} - \hat{p}^i\,\hat{p}^j), \tag{2.8}$$

and

$$\int \frac{d\Omega}{4\pi}\,\frac{v^i\,v^j\,v^k}{(\omega - \mathbf{v}\cdot\mathbf{p})^2} \;=\; \frac{c(\omega/p)}{p^2}\,\hat{p}^i\,\hat{p}^j\,\hat{p}^k \;+\; \frac{d(\omega/p)}{p^2}\,(\delta^{jk}\,\hat{p}^i \;+\; \delta^{ik}\,\hat{p}^j \;+\; \delta^{ij}\,\hat{p}^k), \tag{2.9}$$

where

$$a(u) \equiv 1 \;+\; \frac{u^2}{u^2-1} \;-\; 2Q(u), \qquad b(u) \equiv Q(u) \;-\; 1, \tag{2.10}$$

and

$$\begin{aligned} d(u) \;&\equiv\; \frac{3}{2}\,u\,\Big(Q(u) \;-\; 1\Big) \;-\; \frac{1}{2u}\,Q(u), \\ c(u) \;+\; 3\,d(u) \;&\equiv\; \frac{u}{u^2-1}\Big[1 \;-\; 3\,(u^2-1)\,\Big(Q(u) \;-\; 1\Big)\Big], \end{aligned} \tag{2.11}$$

with $Q(u)$ given by (2.7). For $u > 1$, the functions $a(u)$ and $b(u)$ are both positive. By using these results, together with eq. (2.2), we get

$$T^{00}(x) \;=\; \frac{1}{2}\Big(\mathbf{E}^a\cdot\mathbf{E}^a \;+\; \mathbf{B}^a\cdot\mathbf{B}^a\Big) \;+\; \frac{3}{2}\,\omega_p^2\,\frac{\omega^2}{p^2}\Big\{a(\omega/p)\Big(\mathcal{A}_a^L \;-\; \frac{p}{\omega}\,\mathcal{A}_a^0\Big)^2 \;+\; b(\omega/p)\,\boldsymbol{\mathcal{A}}_a^T\cdot\boldsymbol{\mathcal{A}}_a^T\Big\}, \tag{2.12}$$

and

$$\begin{aligned} \mathbf{S}(x) \;=\; \mathbf{E}^a\times\mathbf{B}^a \;&+\; \frac{3}{2}\,\omega_p^2\,\frac{\omega^2}{p^2}\Big\{\Big(c(\omega/p) \;+\; 3\,d(\omega/p)\Big)\Big(\mathcal{A}_a^L \;-\; \frac{p}{\omega}\,\mathcal{A}_a^0\Big)^2\,\hat{\mathbf{p}} \\ &+\; d(\omega/p)\,\boldsymbol{\mathcal{A}}_a^T\cdot\boldsymbol{\mathcal{A}}_a^T\,\hat{\mathbf{p}} \;+\; 2\,d(\omega/p)\Big(\mathcal{A}_a^L \;-\; \frac{p}{\omega}\,\mathcal{A}_a^0\Big)\,\boldsymbol{\mathcal{A}}_a^T\Big\}. \end{aligned} \tag{2.13}$$

In the rest of this letter, we work in the covariant gauge $p_\mu A_a^\mu = 0$. With the Ansatz (2.1), the gauge field equations (1.1) become then

$$p_\nu p^\nu\,\ddot{\mathcal{A}}^\mu \;+\; i\,g\,p^\mu\Big[\mathcal{A}_\nu, \dot{\mathcal{A}}^\nu\Big] \;-\; g^2\Big[\mathcal{A}_\nu, \Big[\mathcal{A}^\nu, \mathcal{A}^\mu\Big]\Big] \;=\; j^\mu. \tag{2.14}$$

(Throughout, the overdots indicate derivatives with respect to the argument of the function, here z.) When contracted with p_μ, eq. (2.14) reduces to

$$\Big[\mathcal{A}_\nu, \dot{\mathcal{A}}^\nu\Big] \,=\, 0, \tag{2.15}$$

(recall that $p^\nu p_\nu > 0$ and $p^\mu j^a_\mu = 0$). Therefore, we need only consider the simplified equation

$$p_\nu p^\nu \,\ddot{\mathcal{A}}^\mu \,-\, g^2 \Big[\mathcal{A}_\nu, \Big[\mathcal{A}^\nu, \mathcal{A}^\mu\Big]\Big] \,=\, j^\mu, \tag{2.16}$$

together with the constraint (2.15). In the present gauge, $\omega\mathcal{A}^0_a = p\mathcal{A}^L_a$, and the induced current (2.3)–(2.4) can be rewritten as

$$\begin{aligned} \rho_a &\;=\; -\,\Omega_L^2\,\mathcal{A}^0_a, \\ \mathbf{j}_a &\;=\; -\,\Omega_L^2\,\mathcal{A}^L_a\,\hat{\mathbf{p}} \,-\, \Omega_T^2\,\boldsymbol{\mathcal{A}}^T_a, \end{aligned} \tag{2.17}$$

where

$$\Omega_L^2 \;\equiv\; 3\,\omega_p^2\,\frac{\omega^2 - p^2}{p^2}\left(Q\Big(\frac{\omega}{p}\Big) \,-\, 1\right), \tag{2.18}$$

and

$$\Omega_T^2 \;\equiv\; \frac{3}{2}\,\omega_p^2\frac{\omega^2}{p^2}\left(1 \,-\, \frac{\omega^2 - p^2}{\omega^2}Q\Big(\frac{\omega}{p}\Big)\right). \tag{2.19}$$

Since $\omega > p$ by assumption, the r.h.s.'s of eqs. (2.18) and (2.19) are both positive, so that Ω_L and Ω_T are real functions of ω/p.

Any solution of the mean field equations (2.15)–(2.19) depends parametrically on the wave-vector p^μ, i.e. $\mathcal{A}^\mu_a = \mathcal{A}^\mu_a(p\cdot x; p_\nu)$. It can be easily verified that $\mathcal{A}^\mu_a$ is homogeneous in p^μ of degree zero, i.e., $\mathcal{A}^\mu_a(\lambda p\cdot x; \lambda p_\nu) = \mathcal{A}^\mu_a(p\cdot x; p_\nu)$ for arbitrary constant λ. Therefore, the solution depends only on three parameters, for instance, p^i/ω, with $i = 1,\,2,\,3$.

At this stage, it is convenient to introduce three polarization vectors $\epsilon^\mu(p;s)$, with $s = 1,\,2,\,3$ and $p_\mu\epsilon^\mu(p;s) = 0$. We choose the vectors $\epsilon^\mu(p; s = 1,\,2)$ transverse to $\hat{\mathbf{p}}$, i.e.,

$$\epsilon^\mu(p;s) \;=\; (0, \mathbf{e}_s(\mathbf{p})), \qquad \mathbf{p}\cdot\mathbf{e}_s(\mathbf{p}) = 0, \quad \mathbf{e}_s(\mathbf{p})\cdot\mathbf{e}'_s(\mathbf{p}) = \delta_{ss'}, \tag{2.20}$$

while

$$\epsilon^\mu(p;3) \;=\; \frac{1}{\sqrt{\omega^2-p^2}}\,(p,\omega\hat{\mathbf{p}}). \tag{2.21}$$

The normalization is such that $\epsilon(p;s)\cdot\epsilon(p;s') \;=\; -\,\delta_{ss'}$. The decomposition of $\mathcal{A}^\mu_a$ on the vectors $\epsilon^\mu(p;s)$ reads

$$\mathcal{A}^\mu_a(z) \;=\; \sum_{s=1}^{3}\epsilon^\mu(p;s)\,\phi^s_a(z), \tag{2.22}$$

where $\phi^s_a(z)$ are new unknown functions. In terms of these functions, the constraint (2.15) reduces to ($\phi^s \equiv \phi^s_a\,T^a$)

$$\sum_{s=1}^{3}\Big[\phi^s,\dot{\phi}^s\Big] \;=\; 0. \tag{2.23}$$

This is satisfied, in particular, by field configurations of the form $\phi^s_a(z) \;=\; \mathcal{O}^s_a\,h_s(z)$ (no summation over s), with constant $\mathcal{O}^s_a$ and arbitrary functions $h_s(z)$. Indeed, for such fields, the color vectors $\{\phi^s_a(z)\}$ and $\{\dot{\phi}^s_a(z)\}$ (s = 1, 2, 3) are parallel in color space for any s. In this letter we restrict ourselves to such configurations for the color group $SU(2)$ ($f^{abc} = \epsilon^{abc}$), and assume, for simplicity, that $\mathcal{O}^s_a = \delta^s_a$, i.e.,

$$\mathcal{A}^\mu(z) \;=\; \sum_{a=1}^{3}\epsilon^\mu(p;a)\,h_a(z)\,T^a. \tag{2.24}$$

The functions $h_a(z)$ (a = 1, 2, 3) satisfy

$$\Big\{(\omega^2-p^2)\,\ddot{h}_a \;+\; g^2\Big(\sum_{b\neq a}h_b^2\Big)\,h_a\Big\}\,\epsilon^\mu(p;a) \;=\; j^\mu_a, \tag{2.25}$$

as follows from eqs. (2.16). By contracting these equations with $\epsilon_\mu(p;a)$ and using eqs. (2.17), we get

$$\begin{aligned}
(\omega^2-p^2)\,\ddot{h}_1 \;+\; \Omega_T^2\,h_1 \;+\; g^2\,(h_2^2+h_3^2)\,h_1 &\;=\; 0,\\
(\omega^2-p^2)\,\ddot{h}_2 \;+\; \Omega_T^2\,h_2 \;+\; g^2\,(h_1^2+h_3^2)\,h_2 &\;=\; 0,\\
(\omega^2-p^2)\,\ddot{h}_3 \;+\; \Omega_L^2\,h_3 \;+\; g^2\,(h_1^2+h_2^2)\,h_3 &\;=\; 0.
\end{aligned} \tag{2.26}$$

This system admits the following integral of the motion,

$$\mathcal{H} \;=\; \frac{1}{2}\,(\omega^2-p^2)\sum_s\dot{h}_a^2+\frac{1}{2}\,\Omega_T^2(h_1^2+h_2^2)+\frac{1}{2}\Omega_L^2h_3^2+\frac{g^2}{2}\,\Big(h_1^2\,h_2^2+h_1^2\,h_3^2+h_2^2\,h_3^2\Big), \tag{2.27}$$

which acts as an effective Hamiltonian for a point particle with coordinates h_a (the corresponding conjugate momenta being $\dot{h}_a$). The potential in (2.27) is a positive, strictly increasing function of the coordinates h_a. Accordingly, the energy conservation prevents any trajectory $\{h_a(z)\}$ from getting too far away from the origin.

The limiting case $p = 0$, $\omega \neq 0$, of the system (2.26), corresponding to global color excitations of the plasma, has been investigated in Refs. [5, 6]. In this limit, $\Omega_L = \Omega_T = \omega_p$, and the system becomes symmetric with respect to permutations of the functions h_a. It is then straigthforward to analytically construct particular periodic solutions which take advantage of this symmetry. For example, the $p = 0$ system admits in-phase periodic oscillations of the type $h_1 = h_2 = h_3$. More generally, numerical studies of the symmetric system[10] showed that the oscillations are quasi-periodic for small amplitude, but they become unstable as the amplitude is increased. We expect these general properties to remain valid for the asymmetric ($\Omega_T \neq \Omega_L$) system (2.26) as well, and, in particular, the small amplitude abelian-like plasma waves to be stable with respect to nonlinear effects. Note that, since the frequencies Ω_T and Ω_L are generally incomensurable, we do not expect to find any periodic solution superposing *both* longitudinal and transverse waves. However, periodic solutions of the system (2.26) corresponding either to longitudinally, or to transversally polarized plane waves can be constructed.

3 Longitudinal plane waves

We set $h_1 = h_2 = 0$ in eqs. (2.26). We get then a simple harmonic oscillator equation for h_3,

$$(\omega^2 - p^2)\, \ddot{h}_3 \,+\, \Omega_L^2\, h_3 \,=\, 0, \tag{3.1}$$

which has the general solution $h_3(z) = C_1 \cos \nu_L z + C_2 \sin \nu_L z$, with $\nu_L \equiv \Omega_L/\sqrt{\omega^2 - p^2}$ and C_1, C_2 are integration constants. It is convenient here to replace $p^\mu = (\omega, \mathbf{p})$ by $k^\mu \equiv \nu_L\, p^\mu = (k^0, \mathbf{k})$, and to set $h_L(k \cdot x) \equiv h_3(z)$. Then, $\Omega_L^2 = k^\mu k_\mu$ and h_L satisfies

$$\ddot{h}_L \,+\, h_L \,=\, 0. \tag{3.2}$$

(The dots refer here to derivation with respect to $k \cdot x$.) With these new notations, the longitudinal plane-waves that we consider are of the form

$$A_L^\mu(x) \,=\, \epsilon^\mu(k;3)\, h_L(k\cdot x)\, T^3, \tag{3.3}$$

where the frequency k^0 is related to the wave vector $k \equiv |\mathbf{k}|$ by

$$k^2 \,=\, 3\,\omega_p^2 \left(Q\Big(\frac{k^0}{k}\Big) \,-\, 1\right). \tag{3.4}$$

This last equation, which is easily deduced from (2.18), is identical to the dispersion equation of the abelian-like longitudinal modes[1]. This is not surprising since the Ansatz (3.3) led us effectively to a linear, abelian-like, problem. In fact, this property holds for any purely longitudinal plane-wave. Indeed, in that case, $\mathcal{A}_a^\mu(z) = \epsilon^\mu(p;3)\,\phi_a^L(z)$, so that $[\mathcal{A}^\mu, \mathcal{A}^\nu] \,=\, 0$ for any pair (μ,ν) and the general equations (2.16) reduce to harmonic equations for each of the functions $\phi_a^L(z)$. The only manifestation of the non-abelian structure lies in the constraint (2.15), which becomes $[\phi^L, \dot\phi^L] = 0$. A particular solution is $\phi_a^L(z) \,=\, q_a\, h_L(z)$, with constant q_a's. For $SU(2)$, this is the only solution, and it is equivalent to the Ansatz (3.3), up to a trivial global color rotation (sending the color vector $q_a\, T^a$ onto the T^3 axis). For a larger color group, other solutions will exist; e.g., for $SU(3)$ we can choose $\phi^L(z) = \phi_3^L(z)\, T^3 \,+\, \phi_8^L(z)\, T^8$, where T^3 and T^8 are the commuting generators of the group.

To complete our analysis of the longitudinal modes, we evaluate the corresponding energy density and Poynting vector. We return to the particular configuration (3.3) and denote by $k^0 = \omega_l(k)$ the solution of the dispersion equation (3.4)[1] ($\omega_l(k) > k$ for any k, $\omega_l(0) = \omega_p$). The associated field strengths are $\mathbf{E}_L(x) \,=\, -\hat{\mathbf{k}}\,\sqrt{\omega_l^2(k) - k^2}\,\dot h_L\, T^3$ and $\mathbf{B}_L \,=\, 0$. By using these results, together with eqs. (1.6)–(1.7), (2.12)–(2.13), and the dispersion relation (3.4), we obtain

$$T_L^{00}(x) \,=\, \frac{1}{2}\Big(\omega_l^2(k) - k^2\Big) \left\{ \dot h_L^2 \,+\, \Big(\frac{3\omega_p^2}{\omega_l^2(k) - k^2} \,-\, 2\Big)\, h_L^2 \right\}, \tag{3.5}$$

and

$$\mathbf{S}_L(x) \,=\, \frac{3}{2}\,\frac{\omega_l(k)}{k} \Big\{ \omega_p^2 \,-\, \Big(\omega_l^2(k) - k^2\Big) \Big\} h_L^2\, \hat{\mathbf{k}}. \tag{3.6}$$

As expected, these quantities satisfy the conservation law

$$\partial_0 T_L^{00}(x) + \partial_i S_L^i(x) = 0, \tag{3.7}$$

for h_L satisfying (3.2). This last equation also clarifies the physical interpretation of the conserved quantity $\mathcal{H}$, eq. (2.27). Indeed, since the fields depend on x^μ only through $k\cdot x$, it follows that $\partial_0 T_L^{00} = \omega_l(k)\,\dot{T}_L^{00}$ and $\partial_i S_L^i = -\mathbf{k}\cdot\dot{\mathbf{S}}_L = -(k/\omega_l(k))\partial_0\,\hat{\mathbf{k}}\cdot\mathbf{S}_L$. Thus, the conservation law (3.7) shows that the following quantity

$$T_L^{00}(x) - \frac{k}{\omega_l(k)}\,\hat{\mathbf{k}}\cdot\mathbf{S}_L(x) = \frac{1}{2}\Big(\omega_l^2(k) - k^2\Big)\Big(\dot{h}_L^2 + h_L^2\Big) \tag{3.8}$$

is constant along the trajectory. This is precisely the integral of the motion $\mathcal{H}_L$, as obtained from eq. (2.27) in which we set $h_1 = h_2 = 0$ and $\Omega_l^2 = \omega_l^2(k) - k^2$.

4 Transverse plane waves

We turn now to the more interesting case of purely transverse plane waves, that is, we consider the system (2.26) for $h_3 = 0$. The resulting two equations,

$$\begin{aligned}(\omega^2 - p^2)\,\ddot{h}_1 + \Omega_T^2\,h_1 + g^2\,h_2^2\,h_1 &= 0,\\ (\omega^2 - p^2)\,\ddot{h}_2 + \Omega_T^2\,h_2 + g^2\,h_1^2\,h_2 &= 0,\end{aligned} \tag{4.1}$$

are symmetric in h_1 and h_2. Again, it is convenient to define $\nu_T \equiv \Omega_T/\sqrt{\omega^2 - p^2}$ and $k^\mu \equiv \nu_T\,p^\mu$. Then, $\Omega_T^2 = k^\mu k_\mu$ and from (2.19) one easily obtains a dispersion equation for k^0,

$$k^2 = \frac{3}{2}\,\omega_p^2\left[\frac{k_0^2}{k_0^2 - k^2} - Q\Big(\frac{k^0}{k}\Big)\right], \tag{4.2}$$

which is identical to that of the linear (or abelian) transverse modes[1]. We denote by $\omega_t(k)$ the positive solution ($\omega_t(k) > k$, $\omega_t(0) = \omega_p$). The transverse plane waves that we are considering here are, in these notations, of the form

$$A_T^0 = 0, \qquad \mathbf{A}_T(x) = h_T^1(k\cdot x)\,\mathbf{e}_1\,T^1 + h_T^2(k\cdot x)\,\mathbf{e}_2\,T^2, \tag{4.3}$$

where $\mathbf{e}_a \equiv \mathbf{e}_a(\mathbf{k})$ (recall eq. (2.20)) and $h_T^a(k \cdot x) \equiv h_a(z)$, for $a = 1, 2$. The corresponding field strengths are $\mathbf{E}_T(x) = -\,\omega_t(k) \sum_{a=1,2} \dot{h}_T^a \, \mathbf{e}_a \, T^a$, and $\mathbf{B}_T(x) = k\Big(\dot{h}_T^2 \, \mathbf{e}_1 \, T^2 - \dot{h}_T^1 \, \mathbf{e}_2 \, T^1\Big) + g \, h_T^1 \, h_T^2 \, \hat{\mathbf{k}} \, T^3$, the overdots denoting here derivatives with respect to $k \cdot x$. Remark that the three chromoelectric vectors $\mathbf{E}_a$ ($a = 1, 2, 3$) are mutually orthogonal, and the same is true for the three chromomagnetic vectors $\mathbf{B}_a$ (of course, $\mathbf{E}_3 = 0$). Furthermore, the electric and the magnetic color vectors are orthogonal, $\mathbf{E}_a \cdot \mathbf{B}_a = 0$ for any color a. This is very much similar to the usual transverse electromagnetic waves.

The energy density and the Poynting vector corresponding to transverse plane-waves are obtained by inserting the expressions for the field potentials and strengths above in eqs. (1.6)–(1.7), (2.12)–(2.13), and using the dispersion relation (4.2):

$$\begin{aligned} T_T^{00}(x) &= \frac{1}{2}\Big(\omega_t^2(k) + k^2\Big)\left[(\dot{h}_T^1)^2 + (\dot{h}_T^2)^2\right] + \frac{g^2}{2}\,(h_T^1)^2\,(h_T^2)^2 \\ &+ \frac{\omega_t^2(k)}{2}\left(\frac{3\omega_p^2}{\omega_t^2(k) - k^2} - 2\right)\Big[(h_T^1)^2 + (h_T^2)^2\Big] \end{aligned} \tag{4.4}$$

and

$$\mathbf{S}_T(x) = \mathbf{k}\,\omega_t(k)\left\{(\dot{h}_T^1)^2 + (\dot{h}_T^2)^2 + \frac{1}{2}\left[1 + 3\,\frac{\omega_t^2(k)}{k^2}\Big(\frac{\omega_p^2}{\omega_t^2(k) - k^2} - 1\Big)\right]\Big[(h_T^1)^2 + (h_T^2)^2\Big]\right\}. \tag{4.5}$$

One easily verifies that

$$T_T^{00}(x) - \frac{k}{\omega_t(k)}\,\hat{\mathbf{k}} \cdot \mathbf{S}_T(x) = \mathcal{H}_T, \tag{4.6}$$

where the integral of the motion $\mathcal{H}_T$ is obtained by setting $h_3 = 0$ in eq. (2.27).

In order to look for solutions of the system (4.1), we define $h_T^a(k \cdot x) \equiv (\Omega_T/g) f_a(k \cdot x)$, $a = 1, 2$, and get the parameter-free system

$$\begin{aligned} \ddot{f}_1(x) + \left[1 + \Big(f_2(x)\Big)^2\right] f_1(x) &= 0, \\ \ddot{f}_2(x) + \left[1 + \Big(f_1(x)\Big)^2\right] f_2(x) &= 0, \end{aligned} \tag{4.7}$$

for the dimensionless functions f_1 and f_2. The integral of the motion $\mathcal{H}_T$ reads

$$\mathcal{H}_T = \frac{\Omega_T^4}{g^2}\,\frac{1}{2}\Big(\dot{f}_1^2 + \dot{f}_2^2 + f_1^2 + f_2^2 + f_1^2\, f_2^2\Big) \equiv \frac{\Omega_T^4}{g^2}\,\theta^2, \tag{4.8}$$

where $\Omega_T^2 = \omega_t^2(k) - k^2$ and θ is a dimensionless parameter. The Hamiltonian $\mathcal{H}_T$, to within the factor Ω_T^4/g^2, is that of a system of two nonlinearly coupled harmonic oscillators with coordinates f_1 and f_2 and total energy θ^2. The system (4.7) has been already analyzed in Refs. [6, 5], where analytic periodic solutions were constructed, as well as in Refs. [10], where the transition from regular to stochastic motion was investigated numerically.

The simplest solutions correspond to one-dimensional harmonic oscillations along the space-color axes 1 or 2. For instance, if $f_2 = 0$, then f_1 satisfies $\ddot{f}_1 + f_1 = 0$, with the general solution $f_1 = a_1 \cos k \cdot x + a_2 \sin k \cdot x$, and $a_1^2 + a_2^2 = 2\theta^2$. This is time periodic, with period $\mathcal{T}_0 = 2\pi/\omega_t(k)$.

Non-linear, but still one-dimensional, periodic solutions are analytically obtained with the Ansatz $f_1 = \pm f_2 \equiv f$. These solutions describe in or out of phase oscillations along the space-color directions 1 and 2. The function $f(k \cdot x)$ satisfies the nonlinear equation

$$\ddot{f} + f + f^3 = 0. \tag{4.9}$$

A particular solution to (4.9) is

$$f(x) = f_\theta \, \mathrm{cn}\Big((2\theta^2+1)^{1/4}\, k \cdot x; \kappa\Big), \tag{4.10}$$

corresponding to the initial conditions $f(0) = f_\theta$ and $\dot{f}(0) = 0$. Here, $\mathrm{cn}(u;\kappa)$ is the Jacobi elliptic cosine of argument u and modulus κ, with

$$\kappa \equiv \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\sqrt{2\theta^2+1}}\right)^{1/2}, \tag{4.11}$$

and the amplitude f_θ is related to the conserved quantity θ by $f_\theta = \left(\sqrt{2\theta^2+1} - 1\right)^{1/2}$. The solution (4.10) is time periodic, with period

$$\mathcal{T} = \frac{4}{\omega_t(k)} \frac{1}{(2\theta^2+1)^{1/4}} K(\kappa), \tag{4.12}$$

where $K(\kappa)$ is the complete elliptic integral of modulus κ. Since $|\mathbf{A}_T| \lesssim T$, $\theta \lesssim 1$, and $\mathcal{T}_\theta$ remains of order of $\mathcal{T}_0 \equiv 2\pi/\omega_t(k)$.

5 Conclusions

We have studied here the generalized Yang-Mills equations in the hot quark-gluon plasma for particular plane-wave fields. For the color group $SU(2)$ and for an appropriate Ansatz showing correlations between coordinate and color spaces, the problem reduces to an effective mechanical system, with three degrees of freedom and nonlinear couplings. Generally, this system describes superpositions of longitudinal and transverse plane waves, with components along all the three color directions. Simple periodic solutions were obtained analytically for the particular case where the longitudinal and the transverse plane waves decouple. In contrast to what happens at zero temperature, here the plane-wave solutions cannot be obtained by simply performing boost transformations on the corresponding solutions in the comoving frame[8]. This is so because the mean field equations and their solutions are written in the rest frame of the plasma.

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